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Conditional Saddle-Point Approximations for Truncated Bivariate Compound Distributions

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Abstract. The majority of existing research that is related to our study aims to explain phenomena in various fields of application that rely on bivariate random variables. Although these distributions have attracted some attention in the literature, little research exists on the bivariate compound distribution due to computational difficulties in implementing it. This study introduces the conditional saddle-point approximation method to the bivariate compound distribution in continuous and discrete settings, which is more powerful than other approximation methods. We discuss conditional approximations for cumulative distribution functions of bivariate compound distributions. Furthermore, examples of continuous and discrete distributions from the bivariate compound truncated Poisson compound class are presented, and comparisons between saddle-point approximations and exact calculations show the high accuracy of the saddle-point methods.

Keywords: Conditional Saddle-point Approximation; Bivariate Compound Distribution; Cumulative Distribution Function; Bivariate Compound Poisson–Gamma Distribution; Bivariate Compound Poisson-Negative Binomial Distribution.

1. Introduction

Sums of random variables of the form $X_1 + X_2 + X_3 + \cdots + X_N$ can be found in a variety of contexts and have a random index N that is independent of the X_I terms. These types of sums are referred to as “stopped sums” by [1] and [2]. Compound distributions have a wide range of applications in insurance claim modelling [3], particle counters, birth processes, shot noise, damage processes, renewal processes [4] [5], branching processes [6], risk theory [7] and stopped random walks [8]. The total claim amount submitted to an insurance provider, where N is the number of claims and the X_i terms are the individual claims that are considered independent of each other, can be modelled using a compound distribution.

The Poisson distribution is one of the most common families among compound distributions, where N is a random variable from $Poisson(\lambda)$. Damage process distributions, such as $Poisson(\lambda) \vee Bernoulli(p)$; the Hermite distribution, $Poisson(\lambda) \vee Binomial(2, p)$; and the Neyman type A distribution, $Poisson(\lambda) \vee Poisson(\phi)$, are all members of the compound Poisson distribution family (see [9]).

The saddle-point method that is used to derive asymptotic approximations of integrals of a certain type is known to give remarkably good approximations. In Daniel’s work [10], it was shown that this technique could be applied to the problem of approximating densities of sums of independent random variables. To apply the resulting approximation, it is necessary to know the cumulant generating function (CGF). A saddle-point approximation for a cumulative distribution function (CDF) was suggested by [11] for continuous distributions. Daniel [12] also introduced two continuity modifications for tail approximation. Robinson [13] presented a general saddle-point approximation technique that can be applied to tail probability approximation. Wang [14] generalized Lugannani and Rice's method to the case of a bivariate probability distribution function using variable transformations. Barndorff-Nielsen [15] derived a saddle-point density approximation for conditional distributions. Skovgaard [16] proposed a saddle-point approximation for conditional distributions. Saddle-point approximations in randomization theory were discussed and developed by [17] and [18]. Butler's work [19] on saddle-point approximations provides a good review of the field and outlines applications of saddle-point approximations.

Most current research focuses on explaining phenomena across diverse application areas that involve bivariate random variables [20] [21]. Let $\boldsymbol{Z}_i$ be a sequence of iid bivariate random vectors, wherein $\boldsymbol{Z}_i = (X_i, Y_i)^T$ with X_i and Y_i possibly being dependent. The bivariate compound distribution is defined by the following distribution:

$$S_N = \boldsymbol{Z}_1 + \boldsymbol{Z}_2 + \boldsymbol{Z}_3 + \cdots + \boldsymbol{Z}_N = \left(\sum_{i=1}^{N} \boldsymbol{X}_i, \sum_{i=1}^{N} \boldsymbol{Y}_i\right)^T, \tag{1}$$

where N is independent of the $\boldsymbol{Z}_{\text{i}}$ terms.

Let $K_{S_N}(t,s) = \log\left(M_{S_N}(t,s)\right)$ be the joint CGF of S_{N}, where $M_{S_{\text{N}}}(t,s)$ is its joint moment-generating function (MGF). It is easy to show, using [22], that the joint CGF of S_N is

$$K_{S_N}(t,s) = K_N\left(K_Z(t,s)\right). \tag{2}$$

Saddle-point approximations can be obtained for any statistic that admits a CGF in (2). Saddle-point approximation works by inverting the CGF to approximate the CDF of S_{N}.

Section 2 presents the main idea of this study and the conditional saddle-point formulas that are used in the following sections. The bivariate compound Poisson–gamma distribution is considered in Sec. 3 as a continuous example of the bivariate compound Poisson class. In Sec. 4, we give an approximation for the bivariate compound Poisson-negative binomial distribution with three continuity corrections. Conclusions are given in Sec. 5.

2. Conditional Saddle-point Approximations for the Bivariate Compound Distribution

We derive saddle-point approximations for conditional densities and mass functions using two saddle-point approximations: one for the joint density and the other for the marginal. Despite appearing to be very complex to present or not feasible, these conditional probability approximations are very important because they give us alternative methods of computation, possibly based upon simulation methods. See Reid [23] for additional details on conditioning techniques in statistical inference.

By approximating the numerator and denominator separately with saddle-point expansion, conditional density can be expanded with large deviation with ease. For more details on the so-called double saddle-point approximation, see [17]. It is evident that this approximation maintains the same characteristics in terms of relative error uniformly within sets of significant deviations as single saddle-point expansions. This method is easy to use if the conditional CGF is tractable for additional calculations, though this is not always the case. However, it should be kept in mind as a preferred choice whenever possible because the expansion calculated below is based on a saddle-point expansion of a multivariate integral, which may be less accurate than for one-dimensional integrals. For the calculation of the expansion for the conditional distribution function, see [18]. The CGF for the distribution of the entire random variable vector under study must be known, and two saddle-point equations need to be solved: one for this random variable vector and one for the vector of conditioning coordinates.

Given that $F(y|x)$ admits a density and that Y is a continuous variable, the estimate provided by [18] can be summarized as follows:

$$\widehat{Pr}(\, Y \geq y \mid X = x \,) \approx 1 - \Phi(\hat{a}) + \phi(\hat{a})\left(\frac{1}{\hat{a}} - \frac{1}{\hat{b}}\right), \qquad \hat{s} \neq 0, \tag{3}$$

where

$$\hat{a} = sgn(\hat{s})\sqrt{2\left[\{K_N(K_Z(\hat{t},0)) - \hat{t}_0 x\} - \{K_N(K_Z(\hat{t},\hat{s})) - \hat{t}x - \hat{s}y\}\right]},$$

$$\hat{b} = \hat{s}\sqrt{\left|K_N''(K_Z(\hat{t},\hat{s}))(K_Z(\hat{t},\hat{s}))^2 + K_Z''(\hat{t},\hat{s})K_N'(K_Z(\hat{t},\hat{s}))\right| / K_{tt}''(\hat{t}_0,0)}$$

and $\hat{t}, \hat{s}$, and $\hat{t}_0$ are the solutions of the saddle-point equations $K'_N(K_Z(\hat{t},\hat{s}))K_Z^{(t)}(\hat{t},\hat{s}) = x$, $K'_N(K_Z(\hat{t},\hat{s}))K_Z^{(s)}(\hat{t},\hat{s}) = y$ (these are the numerator saddle-point in the approximation) and $K'_N(K_Z(\hat{t}_0,0))K_Z^{(t)}(\hat{t}_0,0) = x$ (this is the denominator saddle-point), respectively, where $K_Z^{(t)} = \frac{\partial K_Z}{\partial t}, K_Z^{(s)} = \frac{\partial K_Z}{\partial s}$, and $sgn(\hat{s})$ captures the $sign \pm$ for $\hat{s}$. As long as (x,y) ∈ I_χ(denotes the interior of the convex hull of the support χ), Equation(3) holds. Both sets of saddle-points are located within S, the joint convergence region connected to $K(t,s)$, so both Hessians in ŝ are positive definite. In addition, the square roots of $\hat{s}$ are clearly defined.

The lattice distribution is a discrete probability distribution that is focused on a set of points of the form $\gamma + nh$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$; γ is a real number; and $h > 0$. The lattice distribution step is denoted by the number h. If the support of Y is the integer lattice, then the continuity corrections to (3), as implemented in [18], should be used to achieve the highest accuracy.

First Continuity Correction

If $\hat{t}, \hat{s}$ is the result associated with $K'_N(K_Z(\hat{t},\hat{s}))K'_Z(\hat{t},\hat{s}) = (j,k)$ for a numerator saddle-point whose $\hat{s} \neq 0$, then we have the following:

$$\widehat{Pr}_1(\, Y \geq k \mid X = j \,) \approx 1 - \Phi(\hat{a}) + \phi(\hat{a})\left(\frac{1}{\hat{a}} - \frac{1}{\hat{b}_1}\right), \qquad \hat{s} \neq 0, \tag{4}$$

where

$$\hat{a} = sgn(\hat{s})\sqrt{2\left[\{K_N(K_Z(\hat{t},0)) - \hat{t}_0 j\} - \{K_N(K_Z(\hat{t},\hat{s})) - \hat{t}j - \hat{s}k\}\right]}, \tag{5}$$

$$\hat{b}_1 = \left(1 - e^{-\hat{s}}\right)\sqrt{\left|K_N''(K_Z(\hat{t},\hat{s}))(K_Z(\hat{t},\hat{s}))^2 + K_Z''(\hat{t},\hat{s})K_N'(K_Z(\hat{t},\hat{s}))\right| / K_{tt}''(\hat{t}_0,0)}\,. \tag{6}$$

Therefore, $\hat{t_0}$ solves $K'_N(K_Z(\hat{t}_0,0))K_Z^{(t)}(\hat{t}_0,0) = j$.

Second Continuity Correction

Given that the offset saddle-point is represented by $(\tilde{t}, \tilde{s})$, and $k^- = k - 1/2$ denotes the offset of k, we can compute

$$K'_N\big(K_Z(\hat{t},\hat{s})\big)K_Z(\hat{t},\hat{s}) = \left(j, k-\tfrac{1}{2}\right). \tag{7}$$

Given that $\tilde{s} \neq 0$, then

$$\widehat{Pr}_2(\,Y \geq k \mid X = j\,) \approx 1 - \Phi(\tilde{a}_2) + \phi(\tilde{a}_2)\left(\tfrac{1}{\tilde{a}_2} - \tfrac{1}{\tilde{b}_2}\right), \qquad \tilde{s} \neq 0, \tag{8}$$

where

$$\tilde{a}_2 = sgn(\tilde{s})\sqrt{2\left[\left\{K_N\big(K_Z(\hat{t},0)\big) - \hat{t}_0 j\right\} - \left\{K_N\big(K_Z(\hat{t},\tilde{s})\big) - \tilde{t}j - \tilde{s}k^{-}\right\}\right]}, \tag{9}$$

$$\tilde{b}_2 = 2\,sinh\left(\tfrac{\tilde{s}}{2}\right)\sqrt{\left|K''_N\big(K_Z(\tilde{t},\tilde{s})\big)\big(K_Z(\tilde{t},\tilde{s})\big)^2 + K''_Z(\tilde{t},\tilde{s})K'_N\big(K_Z(\tilde{t},\tilde{s})\big)\right|/K''_{tt}(\hat{t}_0,0)}, \tag{10}$$

and the saddle-point, $\hat{t}_0$, remains unaltered.

3. The Bivariate Compound Poisson-Gamma Distribution

As a continuous distribution example, consider S_N in (1) where N is distributed as $Poisson(\lambda)$ with $M_N(s) = \exp\lambda(e^s - 1)$ and the X_i and Y_i terms are iid random variables that follow a $Gamma(\alpha_1, \beta_1) \vee Gamma(\alpha_2, \beta_2)$ distribution, with a joint distribution $M_Z(t,s) = (1-\beta_1 t)^{-\alpha_1}(1-\beta_2 t)^{-\alpha_2}$ that is independent of N; then S_N has the bivariate compound Poisson–gamma distribution [24]. Applications of the Poisson–gamma model are wide ranging: it has been used on catch and effort data in [25], in recruitment in multicentre trials, in insurance, and in pump failure [26]. As the S_N distribution is not continuous due to a point probability, $Pr(N=0)$, at zero, the truncated distribution can be used [27]. However, the MGF of N is

$$M_N(s) = [\exp\lambda\,(e^s - 1) - \exp(-\lambda)]/(1 - \exp(-\lambda));$$

hence, the MGF of $\widehat{S_N} = \sum_{i=1}^{\widehat{N}} Z_i$ is

$$M_{S_N}(t,s) = M_N\left(log\,M_{X,Y}\,(t,s)\right), \tag{11}$$

$$M_{S_N}(t,s) = \frac{\exp\left(\lambda\big(M_{x,y}(t,s) - 1\big)\right) - \exp(-\lambda)}{(1-\exp(-\lambda))},$$

and the CGF of S_N is

$$K_{\hat{S}_N}(\hat{t},\hat{s}) = \log M_{\hat{S}_N}\,(\hat{t},\hat{s}),$$

$$= \log\left((1-q)^{-1}\left[\exp\left(\lambda\left(\frac{1}{(1-\beta_1\hat{t})^{\alpha_1}}\right)\left(\frac{1}{(1-\beta_2\hat{s})^{\alpha_2}}\right) - \lambda\right) - q\right]\right),$$
$$= \log\left(\frac{\left[\exp\left(\lambda\left(\frac{1}{(1-\beta_1\hat{t})^{\alpha_1}}\right)\left(\frac{1}{(1-\beta_2\hat{s})^{\alpha_2}}\right) - \lambda\right) - q\right]}{(1-q)}\right), \tag{12}$$

where $q = e^{-\lambda}$.

The numerator saddle-point solves

$$K'_N\left(K_Z(\hat{t},\hat{s})\right)K_Z^{(t)}(\hat{t},\hat{s}) = x,$$
$$K'_N\left(K_Z(\hat{t},\hat{s})\right)K_Z^{(s)}(\hat{t},\hat{s}) = y, \tag{15}$$

that is,

$$\frac{\lambda\alpha_1\beta_1\, exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_1 t)\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right]} - x = 0, \tag{13}$$

$$\frac{\lambda\alpha_2\beta_2\, exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_2 s)\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right]} - y = 0, \tag{14}$$

which implies (see Appendix A)

$$\hat{t} = \frac{1}{\beta_1}\left\{1 - \left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}}\right\},$$
$$\hat{s} = \frac{1}{\beta_2}\left[1 - \left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 x}{\alpha_2\beta_2 y}\right)^{\alpha_1}\right]^{\frac{1}{\alpha_1+\alpha_2}}\right], \tag{15}$$

and the denominator saddle-point solves

$$K'_N\left(K_Z(\hat{t}_0,0)\right)K_Z^{(t)}(\hat{t}_0,0) = x, \tag{16}$$

that is,

$$\frac{\lambda\alpha_1\beta_1\, exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}}-\lambda\right)}{(1-\beta_1 t)^{\alpha_1}(1-\beta_1 t)\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}}-\lambda\right)-q\right]} - x = 0, \tag{17}$$

which gives (see Appendix B)

$$\widehat{t_0} = \frac{1}{\beta_1}\left[1 \pm \left(\frac{\lambda}{z}\right)^{\frac{1}{\alpha_1}}\right]. \tag{18}$$

In addition,

$$\begin{aligned} K_t''(t,s) = &\frac{\lambda\alpha_1^2\beta_1^2\, exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_1 t)^2\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right]} \\ &+\frac{\lambda\alpha_1\beta_1^2\, exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_1 t)^2\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right]} \\ &+\frac{\lambda^2\alpha_1^2\beta_1^2\, exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)}{(1-\beta_1 t)^{2\alpha_1}(1-\beta_2 s)^{2\alpha_2}(1-\beta_1 t)^2\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right]} \end{aligned} \tag{19}$$

$$-\frac{\lambda^2\alpha_1^2\beta_1^2\left(exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)\right)^2}{(1-\beta_1 t)^{2\alpha_1}(1-\beta_2 s)^{2\alpha_2}(1-\beta_1 t)^2\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right.}$$

$$K_s''(t,s)=\frac{\lambda\alpha_2^2\beta_2^2\, exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_2 s)^2\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right]}$$
$$+\frac{\lambda\alpha_2\beta_2^2\, exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_2 s)^2\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right]}$$
$$+\frac{\lambda^2\alpha_2^2\beta_2^2\, exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)}{(1-\beta_1 t)^{2\alpha_1}(1-\beta_2 s)^{2\alpha_2}(1-\beta_2 s)^2\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right]} \quad (20)$$
$$-\frac{\lambda^2\alpha_2^2\beta_2^2\left(exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)\right)^2}{(1-\beta_1 t)^{2\alpha_1}(1-\beta_2 s)^{2\alpha_2}(1-\beta_2 s)^2\left[exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}}-\lambda\right)-q\right.}$$

Note that

$$K''_N\big(K_Z(\tilde{t},\tilde{s})\big)\big(K_Z(\tilde{t},\tilde{s})\big)^2+K''_Z(\tilde{t},\tilde{s})K'_N\big(K_Z(\tilde{t},\tilde{s})\big)=K''(t,s).$$

This Hessian matrix can be obtained by

$$K''(t,s)=\begin{bmatrix}K''_{tt}(t,s) & K''_{ts}(t,s)\\ K''_{st}(t,s) & K''_{ss}(t,s)\end{bmatrix}, \qquad K''_{tt}(t,s)=\frac{\partial^2}{\partial t^2}K(t,s), \qquad etc.,$$

with $\det K''(t,s)>0$ (positive definite).

The conditional saddle-point approximation for the continuous CDFs $\widehat{Pr}(Y\geq y \mid X=x)$ given by (3) and $\hat{t},\hat{s}$, and $\hat{t}_0$ are as in (16) and (19), respectively.

To see the accuracy of the conditional saddle-point approximation, consider the $Poisson(4)\vee Gamma(1{,}2), Gamma(1{,}4)$ distribution. Table 1 provides the exact value and conditional saddle-point approximation of $Pr(Y\geq y|X=x)$ at some values of x,y. The exact value of the CDF of S_N is calculated by simulating 10^6 values of S_N. Each value is calculated by generating N, from $Poisson(4)$ and x,y values from the $Gamma(1{,}2), Gamma(1{,}4)$ distribution. The conditional saddle-point approximation is very accurate.

Table 1. The exact and conditional saddle-points of the CDF for the bivariate compound truncated $\boldsymbol{Poisson(4)}\vee$ $\boldsymbol{Gamma(1,2), Gamma(1,4)}$ distribution.

X	Y	Exact	$\hat{F}(y\|x)$	Relative Error
2.5	1.5	0.0677707	0.0689926	0.017710595
2.5	2.0	0.0740187	0.09687894	0.235967074
2.5	3.5	0.0918188	0.1890771	0.514384344
5.5	1.5	0.0766979	0.02113559	2.62885.673
5.5	2.5	0.0975687	0.04549499	1.144603175
7.5	1.5	0.078945	0.01104324	6.148717224
7.5	2.5	0.1025864	0.02580095	2.976070649
7.5	3.5	0.128174	0.04645276	1.759233251

4. **The Bivariate Compound Poisson-negative Binomial Distribution**

The bivariate compound Poisson-negative binomial models are naturally found in the fields of insurance and actuarial science, and many authors have studied them [28]. The issue of approximate compound Poisson distributions for compound negative binomial distributions was addressed by [29], [30], [31], and [32]. Numerous social, financial, and physical issues can be effectively modelled using this distribution – for example, the total number of orders placed and items sold each day, the total number of insurance claims and claimants each hour, the total number of injury accidents and fatalities, and the total number of visits and drugs prescribed.

Let N be the number of claims that occurred in a fixed time period, which follows a $Poisson(\lambda)$ distribution with $M_N(s) = \exp\lambda(e^s - 1)$, where the X_i' and Y_i' terms are iid random variables that follow a $\big(Negative-Binomial(r_1,p_1), Negative-Binomial(r_2,p_2)\big)$ ditribution, with a joint distribution $M(t,s) = \big(p_1/(1-q_1e^t)\big)^{-r_1}\big(p_2/(1-q_2e^s)\big)^{-r_2}$. Then, S_N in (1) follows a bivariate compound Poisson-negative binomial distribution with CGF

$$k_{s_N}(t,s) = \lambda\left(\left(\frac{p_1}{1-q_1e^t}\right)^{r_1}\left(\frac{p_2}{1-q_2e^s}\right)^{r_2} - 1\right). \tag{21}$$

The numerator saddle-point, $\hat{t},\hat{s}$, and the denominator saddle-point, $\hat{t}_0$, solve the saddle-point equations $K'_t(t_0,s_0) = j$, $K'_s(t_0,s_0) = k$ and $K'_t(\hat{t}_0,0) = j$, respectively, for $j,k = 0,1,2,\ldots$, and are given by

$$\tilde{s} = log\left(\frac{1-w}{q_2}\right), \tag{22}$$

$$\tilde{t} = log\left(\frac{1-v}{q_1}\right). \tag{23}$$

The values of $w, v \notin \{0,1\}$, respectively, are found in the following equations (see Appendix C):

$$r_1^{r_1}y^{r_1+1}w^{r_1+r_2+1} = \lambda p_1^{r_1}p_2^{r_2}r_2(1-w)[r_2x + (r_1y - r_2x)w]^{r_1}, \tag{24}$$

$$r_2^{r_2}x^{r_2+1}v^{r_1+r_2 1} = \lambda p_1^{r_1}p_2^{r_2}r_1(1-v)[r_1y + (r_2x - r_1y)v]^{r_2}, \tag{25}$$

and the denominator saddle-point is

$$\hat{t}_0 = log\left\{\frac{1}{q_1}\left[1 - \sqrt[3]{\frac{\lambda p_1^2p_2^{r_2}}{x(1-q_2)^{r_2}} + \sqrt{\frac{\lambda^2p_1^4p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3p_1^6p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}} - \sqrt[3]{\frac{\lambda p_1^2p_2^{r_2}}{x(1-q_2)^{r_2}} - \sqrt{\frac{\lambda^2p_1^4p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3p_1^6p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}\right]\right\},$$

See Appendix D.

When $j = 0$ or $k = 0$, $Pr(N=0) = e^{-\lambda}$. For $j,k \neq 0$, the conditional saddle-point approximation for the distribution is calculated by (5), where $K(\hat{t},\hat{s})$ $K''_{tt}(\hat{t}_0,0), K(\hat{t}_0,0)$, and $|K''(t,s)|$.

Conditional saddle-point continuity-corrected approximations $\widehat{Pr_i} = (Y \geq k|X = j)$, $i =$ 1,2 are given in (5) and (9).

To assess the accuracy of the conditional saddle-point approximations, consider the $Poisson(5) \vee Negative - Binomial(2{,}0.5), Negative - Binomial(2{,}0.5)$ distribution. Table 2 provides the saddle-point approximations to the distribution of S_N. Each line represents the exact value, $\widehat{Pr} = (Y \geq k|X = j)$, with the two continuity-corrections for $j, k = 1, 2, 3, \ldots$ The exact p value is calculated by simulating 10^6 values for S_N by simulating N from $Poisson(5)$ values and generating N values from the $Negative - Binomial(2{,}0.5), Negative - Binomial(2{,}0.5)$ distribution. The saddle-point approximations show great accuracy. In most cases the second correction is better than the first correction. This could be because of the continuity correction term, $(1 - e^{-\hat{s}})$ (Equation 7), which seems to modify the CDF probability in a way that isn't consistent with the continuity correction.

Table 2. The exact and conditional saddle-points of the CDF for the bivariate compound $\boldsymbol{Poisson(5)} \vee \boldsymbol{NegativeBinomial(2, 0.5), NegativeBinomial(2, 0.5)}$ distribution.

X	Y	Exact	$\widehat{Pr}_1$	$\widehat{Pr}_2$	Relative Error (*Pr1*)	Relative Error (*Pr2*)
1	1	0.25	0.2787989	0.2618004	0.103296319	0.045074034
1	2	0.34375	0.40079085	0.3914946	0.142320739	0.121954706
1	3	0.40625	0.51601354	0.5099798	0.212714457	0.203399883
2	1	0.171875	0.18385286	0.1718464	0.065149163	0.000166602
2	2	0.236328	0.29021717	0.2828212	0.185685671	0.164390759
2	3	0.279297	0.4003707	0.3951325	0.302403997	0.293156075
3	1	0.135417	0.12920385	0.1203312	0.048087963	0.125369169
3	2	0.186198	0.21887119	0.212859	0.149280451	0.125251722
3	3	0.220052	0.31871378	0.3141364	0.309562329	0.299501767
4	1	0.111328	0.09403932	0.0873369	0.183845226	0.274695596
4	2	0.153076	0.16889246	0.1639764	0.093648112	0.066475647
4	3	0.180908	0.2574482	0.253468	0.297303302	0.286268737
5	1	0.09375	0.070077	0.064937	0.337814119	0.443706978
5	2	0.128906	0.132383	0.128347	0.026264702	0.00435538
5	3	0.152344	0.210072	0.206624	0.274801021	0.262699396

5. Conclusion

This article provides methods for calculating conditional saddle-point approximations for bivariate compound distributions. The bivariate compound class includes the bivariate Hermite distribution, the bivariate Neyman type A distribution, the bivariate Pólya–Aeppli distribution, and the bivariate Thomas distribution. Our conditional saddle-point approximations are very accurate and quickly calculated approximations. Moreover, our method is easy to implement and requires little computational effort. We demonstrate the effectiveness of our conditional saddle-point approximations in both discrete and continuous settings using numerical examples.

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Appendix A

We solve the saddle-point equations as below:

$$\frac{\lambda\alpha_1\beta_1\exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2}-\lambda\right\}}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_1 t)\left[\exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2}-\lambda\right\}-e^{-\lambda}\right]}-x=0,$$

$$\frac{\lambda\alpha_2\beta_2\exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2}-\lambda\right\}}{(1-\beta_1 t)^{\alpha_1}(1-\beta_2 s)^{\alpha_2}(1-\beta_2 s)\left[\exp\left\{\lambda(1-\beta_1 t)^{-\alpha_1}(1-\beta_2 s)^{-\alpha_2}-\lambda\right\}-e^{-\lambda}\right]}-y=0.$$

The terms x and y are placed on the right side of their respective equations, and the first equation is divided by the second equation to obtain the relation.

$$\frac{\alpha_1\beta_1(1-\beta_2 s)}{\alpha_2\beta_2(1-\beta_1 t)}=\frac{x}{y}. \tag{A.1}$$

The substitutions $(1-\beta_1 t)=w$ and $(1-\beta_2 s)=v$ are made so that equation (A.1) takes the form

$$\frac{\alpha_1\beta_1 v}{\alpha_2\beta_2 w}=\frac{x}{y}.$$

and

$$v=\frac{\alpha_2\beta_2 wx}{\alpha_1\beta_1 y} \tag{A.2}$$

Then, the first equation can be rewritten as

$$\frac{\lambda\alpha_1\beta_1\exp\left(\frac{\lambda}{w^{\alpha_1}\left(\frac{\alpha_2\beta_2 wx}{\alpha_1\beta_1 y}\right)^{\alpha_2}}-\lambda\right)}{w^{\alpha_1}\left(\frac{\alpha_2\beta_2 wx}{\alpha_1\beta_1 y}\right)^{\alpha_2}w\left[\exp\left(\frac{\lambda}{w^{\alpha_1}\left(\frac{\alpha_2\beta_2 wx}{\alpha_1\beta_1 y}\right)^{\alpha_2}}-\lambda\right)-q\right]}=x. \tag{A.3}$$

Making another substitution, $=\frac{\lambda}{w^{\alpha_1}\left(\frac{\alpha_2\beta_2 wx}{\alpha_1\beta_1 y}\right)^{\alpha_2}}$, we obtain

$$z=\frac{\lambda}{w^{\alpha_1}w^{\alpha_2}\left(\frac{\alpha_2\beta_2 x}{\alpha_1\beta_1 y}\right)^{\alpha_2}}=\frac{\lambda}{w^{\alpha_1+\alpha_2}}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2},$$

$$w^{\alpha_1+\alpha_2}=\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2},$$

$$w=\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}}. \tag{A.4}$$

with the caution that a double sign may appear if $\alpha_1+\alpha_2$ is even, Equation (A.3) takes the form

$$\frac{z\alpha_1\beta_1\exp(z-\lambda)}{\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}}[\exp(z-\lambda)-q]}=x.$$

then

$$\frac{z\alpha_1\beta_1\exp(z-\lambda)}{\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}}}=x[\exp(z-\lambda)-q],$$

$$\frac{z\alpha_1\beta_1\exp(z-\lambda)}{\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}}}=x\exp(z-\lambda)-xq,$$

$$\frac{z^{1+\frac{1}{\alpha_1+\alpha_2}}\alpha_1\beta_1\exp(z-\lambda)}{\lambda^{\frac{1}{\alpha_1+\alpha_2}}}\left(\frac{\alpha_2\beta_2 x}{\alpha_1\beta_1 y}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}-x\exp(z-\lambda)+xq=0,$$

$$\frac{(\alpha_1\beta_1)^{1-\frac{\alpha_2}{\alpha_1+\alpha_2}}}{\lambda^{\frac{1}{\alpha_1+\alpha_2}}}\left(\frac{\alpha_2\beta_2 x}{y}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}z^{\frac{\alpha_1+\alpha_2+1}{\alpha_1+\alpha_2}}\exp(z-\lambda)-x\exp(z-\lambda)+xq=0,$$

$$\left[(\alpha_1\beta_1)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}\left(\frac{\alpha_2\beta_2 x}{\lambda^{\frac{1}{\alpha_2}}y}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}z^{\frac{\alpha_1+\alpha_2+1}{\alpha_1+\alpha_2}}-x\right]\exp(z-\lambda)+xq=0,$$

$$\left[(\alpha_1\beta_1)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}\left(\frac{\alpha_2\beta_2 x}{\lambda^{\frac{1}{\alpha_2}}y}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}z^{\frac{\alpha_1+\alpha_2+1}{\alpha_1+\alpha_2}}-x\right]\exp(z)+xq\exp(\lambda)=0, \tag{A.5}$$

which is an equation that implicitly depends only on t and not on s. Again, the caution of a double sign before the coefficient of $z^{\frac{\alpha_1+\alpha_2+1}{\alpha_1+\alpha_2}}$ may appear if $\alpha_1+\alpha_2$ is even. Once a value of z that satisfies equation (A.5) is determined, the corresponding value of t is found by substituting back as follows:

Recall equation (A.4) as

$$w=\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}},$$

and the substitution $(1-\beta_1 t)=w$, we obtain

$$1-\beta_1 t=\left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}}.$$

then

$$\beta_1 t = 1 - \left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}},$$

$$t = \frac{1}{\beta_1}\left\{1 - \left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 y}{\alpha_2\beta_2 x}\right)^{\alpha_2}\right]^{\frac{1}{\alpha_1+\alpha_2}}\right\},$$

with the caution that a double sign may appear before the power if $\alpha_1 + \alpha_2$ is even.

To obtain the value of s, observe that z can also be written in terms of v only, as follows: equation (A.2) can be manipulated to obtain

$$w = \frac{\alpha_1\beta_1 v y}{\alpha_2\beta_2 x};$$

therefore,

$$z = \frac{\lambda}{\left(\frac{\alpha_1\beta_1 v y}{\alpha_2\beta_2 x}\right)^{\alpha_1} v^{\alpha_2}},$$

and as with equation (A.4), we obtain that

$$v = \left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 x}{\alpha_2\beta_2 y}\right)^{\alpha_1}\right]^{\frac{1}{\alpha_1+\alpha_2}},$$

with the caution of a double sign if $\alpha_1 + \alpha_2$ is even. This implies that equation (A.5) can also be seen as an equation that implicitly depends only on s and not on t. Once a value of z is found, the corresponding value of s can be found by substituting backwards:

$$1 - \beta_2 s = \left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 x}{\alpha_2\beta_2 y}\right)^{\alpha_1}\right]^{\frac{1}{\alpha_1+\alpha_2}},$$

$$\beta_2 s = 1 - \left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 x}{\alpha_2\beta_2 y}\right)^{\alpha_1}\right]^{\frac{1}{\alpha_1+\alpha_2}},$$

$$s = \frac{1}{\beta_2}\left[1 - \left[\frac{\lambda}{z}\left(\frac{\alpha_1\beta_1 x}{\alpha_2\beta_2 y}\right)^{\alpha_1}\right]^{\frac{1}{\alpha_1+\alpha_2}}\right],$$

with the caution that a double sign may appear if $\alpha_1 + \alpha_2$ is even.

Appendix B

To find the value of $\hat{t}_0$ we solve

$$\frac{\partial K_1(t)}{\partial t} = x,$$

$$\frac{\lambda\alpha_1\beta_1 \exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}} - \lambda\right)}{(1-\beta_1 t)^{\alpha_1}(1-\beta_1 t)\left[\exp\left(\frac{\lambda}{(1-\beta_1 t)^{\alpha_1}} - \lambda\right) - q\right]} - x = 0.$$

The substitution $z = \frac{\lambda}{(1-\beta_1 t)^{\alpha_1}}$ is applied to obtain

$$1 - \beta_1 t = \left(\frac{\lambda}{z}\right)^{\frac{1}{\alpha_1}}$$

with the caution that a double sign may appear before the power if α_1 is even. The equation is transformed into

$$x = \frac{z\alpha_1\beta_1 \exp(z-\lambda)}{\left(\frac{\lambda}{z}\right)^{\frac{1}{\alpha_1}}[\exp(z-\lambda)-q]},$$

and we have that

$$z^{\frac{\alpha_1+1}{\alpha_1}}\alpha_1\beta_1 \exp(z) = \lambda^{\frac{1}{\alpha_1}}x(\exp(z) - q\exp(\lambda)),$$

$$\left(\alpha_1\beta_1 z^{\frac{\alpha_1+1}{\alpha_1}} - \lambda^{\frac{1}{\alpha_1}}x\right)\exp(z) + \lambda^{\frac{1}{\alpha_1}}xq\exp(\lambda) = 0$$

as in Equation (A.5). Therefore, the solution of the original equation is

$$\widehat{t_0} = \frac{1}{\beta_1}\left[1 - \left(\frac{\lambda}{z}\right)^{\frac{1}{\alpha_1}}\right],$$

where z satisfies

$$\left(\alpha_1\beta_1 z^{\frac{\alpha_1+1}{\alpha_1}} - \lambda^{\frac{1}{\alpha_1}}x\right)\exp(z) + \lambda^{\frac{1}{\alpha_1}}xq\exp(\lambda) = 0.$$

In addition, if α_1 is even, the double-sign versions

$$\left(\pm\alpha_1\beta_1 z^{\frac{\alpha_1+1}{\alpha_1}} - \lambda^{\frac{1}{\alpha_1}}x\right)\exp(z) + \lambda^{\frac{1}{\alpha_1}}xq\exp(\lambda) = 0,$$

$$\widehat{t_0} = \frac{1}{\beta_1}\left[1 \pm \left(\frac{\lambda}{z}\right)^{\frac{1}{\alpha_1}}\right],$$

should be considered.

Appendix C

We solve the saddle-point equations to find t, s as below.

The variable t is eliminated in the equation

$$\lambda\left[\frac{p_1^{r_1}r_1q_1e^t}{(1-q_1e^t)^{r_1+1}}\frac{p_2^{r_2}}{(1-q_2e^s)^{r_2}}\right] - x = 0,$$

and the variable s is eliminated in the equation

$$\lambda\left[\frac{p_1^{r_1}}{(1-q_1e^t)^{r_1}}\frac{p_2^{r_2}r_2q_2e^s}{(1-q_2e^s)^{r_2+1}}\right] - y = 0.$$

This is achieved by substituting $1 - q_1e^t = w$ and $1 - q_2e^s = v$ into the equation to obtain

$$\frac{\lambda p_1^{r_1}p_2^{r_2}r_1(1-v)}{v^{r_1+1}w^{r_2}} - x = 0, \quad \text{(C.1)}$$

$$\frac{\lambda p_1^{r_1}p_2^{r_2}r_2(1-w)}{w^{r_2+1}v^{r_1}} - y = 0. \quad \text{(C.2)}$$

The goal is to eliminate the variable v in Equation (C.1) and the variable w in Equation (C.2). Observe that $w \neq 0$ and $v \neq 0$. The term w^{r_2} in Equation (C.1) is isolated to obtain

$$w^{r_2} = \frac{\lambda p_1^{r_1}p_2^{r_2}r_1(1-v)}{v^{r_1+1}x}, \quad \text{(C.3)}$$

and the term v^{r_1} in equation (C.2) is isolated to obtain

$$v^{r_1} = \frac{\lambda p_1^{r_1} p_2^{r_2} r_2 (1-w)}{w^{r_2+1} y}. \tag{C.4}$$

It is observed that $w \neq 1$ and $v \neq 1$. Equation (C.3) is divided by Equation (C.4) to obtain

$$\frac{w^{r_2}}{v^{r_1}} = \frac{\lambda p_1^{r_1} p_2^{r_2} r_1 (1-v)}{v^{r_1+1} x} \frac{w^{r_2+1} y}{\lambda p_1^{r_1} p_2^{r_2} r_2 (1-w)},$$

$$\frac{w^{r_2}}{v^{r_1}} = \frac{r_1 w^{r_2+1} y (1-v)}{r_2 v^{r_1+1} x (1-w)}.$$

Therefore,

$$\frac{r_1 y w (1-v)}{r_2 x v (1-w)} = 1,$$

which implies that

$$r_1 y w - r_1 y w v = r_2 x v - r_2 x v w.$$

Then,

$$r_1 y w + (r_2 x - r_1 y) w v - r_2 x v = 0,$$

leading to

$$w = \frac{r_2 x v}{r_1 y + (r_2 x - r_1 y) v},$$

and

$$v = \frac{r_1 y w}{r_2 x + (r_1 y - r_2 x) w}. \tag{C.5}$$

In addition,

$$1 - w = \frac{r_1 y (1-v)}{r_1 y + (r_2 x - r_1 y) v},$$

and

$$1 - v = \frac{r_2 x (1-w)}{r_2 x + (r_1 y - r_2 x) w}. \tag{C.6}$$

The values of v in Equation (C.5) and $1 - v$ in Equation (C.6) are substituted into Equation (C.3) to obtain

$$w^{r_2} = \frac{\lambda p_1^{r_1} p_2^{r_2} r_1 \dfrac{r_2 x (1-w)}{r_2 x + (r_1 y - r_2 x) w}}{x \left[\dfrac{r_1 y w}{r_2 x + (r_1 y - r_2 x) w}\right]^{r_1+1}},$$

$$\frac{r_1^{r_1+1} x y^{r_1+1} w^{r_1+r_2 1}}{[r_2 x + (r_1 y - r_2 x) w]^{r_1+1}} = \frac{\lambda p_1^{r_1} p_2^{r_2} r_1 r_2 x (1-w)}{r_2 x + (r_1 y - r_2 x) w},$$

$$\frac{r_1^{r_1} y^{r_1+1} w^{r_1+r_2 1}}{[r_2 x + (r_1 y - r_2 x) w]^{r_1}} = \lambda p_1^{r_1} p_2^{r_2} r_2 (1-w),$$

$$r_1^{r_1} y^{r_1+1} w^{r_1+r_2+1} = \lambda p_1^{r_1} p_2^{r_2} r_2 (1-w) [r_2 x + (r_1 y - r_2 x) w]^{r_1}, \tag{C.7}$$

which is an implicit equation of s that depends on all the original variables except t. In a similar fashion, we obtain the relation

$$r_2^{r_2} x^{r_2+1} v^{r_1+r_2 1} = \lambda p_1^{r_1} p_2^{r_2} r_1 (1-v) [r_1 y + (r_2 x - r_1 y) v]^{r_2}, \tag{C.8}$$

which is an implicit equation of t that depends on all the original variables except s. Once the value of $w \notin \{0,1\}$ is found in equation (F.7), the corresponding value of s is

$$s = \log\left(\frac{1-w}{q_2}\right), \tag{C.9}$$

and once the value of $v \notin \{0,1\}$ is found in Equation (C.8), the corresponding value of t is

$$t = \log\left(\frac{1-v}{q_1}\right). \tag{C.10}$$

The difficulty in finding the values of w and v in Equations (C.7) and (C.8) (respectively, the values of s and t in Equations (C.9) and (C.10) depends on the values of r_1 and r_2. For example, if $r_1 = r_2 = 1$, then Equation (C.7) transforms into

$$y^2w^3 = \lambda p_1p_2(1-w)[x+(y-x)w],$$
$$y^2w^3 + \lambda p_1p_2(y-x)w^2 + \lambda p_1p_2(2x-y)w - \lambda p_1p_2x = 0$$

and Equation (C.8) transforms into

$$x^2v^3 + \lambda p_1p_2(x-y)v^2 + \lambda p_1p_2(2y-x)v - \lambda p_1p_2y = 0.$$

These are polynomial equations of degree 3 that can be algebraically resolved. More generally, if r_1 and r_1 are positive integers, Equations (C.7) and (C.8) can be rewritten, respectively, as

$$P(w) = 0,$$

and

$$Q(v) = 0,$$

where P is a polynomial function of w with coefficients in the original variables excluding t, Q is a polynomial function of v with coefficients in the original variables excluding s and both P and Q are of degree $r_1 + r_2 + 1$.

Appendix D

The second saddle-point equation is

$$\frac{\partial K(t,0)}{\partial t} = x.$$

Hence,

$$\lambda\left[\frac{p_1^{r_1}r_1q_1e^t}{(1-q_1e^t)^{r_1+1}}\frac{p_2^{r_2}}{(1-q_2)^{r_2}}\right] - x = 0.$$

The substitution $1 - q_1e^t = w$ is applied to obtain

$$\frac{\lambda p_1^{r_1}r_1(1-w)p_2^{r_2}}{w^{r_1+1}(1-q_2)^{r_2}} - x = 0$$

and this equation can be manipulated to obtain

$$\frac{\lambda p_1^{r_1}r_1(1-w)p_2^{r_2}}{w^{r_1+1}(1-q_2)^{r_2}} = x,$$
$$\lambda p_1^{r_1}r_1(1-w)p_2^{r_2} = xw^{r_1+1}(1-q_2)^{r_2},$$
$$x(1-q_2)^{r_2}w^{r_1+1} + \lambda p_1^{r_1}p_2^{r_2}r_1w - \lambda p_1^{r_1}p_2^{r_2}r_1 = 0,$$

which is a polynomial in the variable w of degree $r_1 + 1$, namely, one polynomial equation of the type

$$Aw^{r_1+1} + Bw - B = 0, \tag{D.1}$$

with $A = x(1-q_2)^{r_2}$ and $B\ =\ \lambda p\_1\verb|^|\{\{r\_1\}\}p\_2\verb|^|\{\{r\_2\}\}\{r_1\}$. The difficulty of solving this equation by algebraic methods depends on the value of r_1. For example, if $r_1 = 2$, then Equation (D.1) is transformed into

$$Aw^3 + Bw - B = 0, \tag{D.2}$$

which is a depressed cubic

$$x^3 + px + q = 0, \tag{D.3}$$

with $p = \frac{B}{A}$ and $q = -\frac{B}{A}$. $If\, 4p^3 + 27q^2 > 0$, then the depressed cubic (D.3) has the real solution

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

This implies that if $\frac{4B^3}{A^3} + \frac{27B^2}{A^2} > 0$ (for example, if $A > 0$ and $B > 0$), then Equation (D.2) has one real solution

$$w = \sqrt[3]{\frac{B}{2A} + \sqrt{\frac{B^2}{4A^2} + \frac{B^3}{27A^3}}} + \sqrt[3]{\frac{B}{2A} - \sqrt{\frac{B^2}{4A^2} + \frac{B^3}{27A^3}}},$$

which is

$$w = \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} + \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}$$
$$+ \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} - \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}$$

Substituting $w = 1 - q_1 e^t$, we obtain that

$$1 - q_1 e^t = \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} + \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}$$
$$+ \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} - \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}$$

$$q_1 e^t = 1 - \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} + \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}$$
$$- \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} - \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}$$

$$e^t = \frac{1}{q_1}\left[1 - \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} + \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}\right.$$
$$\left. - \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} - \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}\right]$$

$$\hat{t}_0 = \log\left\{\frac{1}{q_1}\left[1 - \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} + \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}\right.\right.$$
$$\left.\left. - \sqrt[3]{\frac{\lambda p_1^2 p_2^{r_2}}{x(1-q_2)^{r_2}} - \sqrt{\frac{\lambda^2 p_1^4 p_2^{2r_2}}{x^2(1-q_2)^{2r_2}} + \frac{8\lambda^3 p_1^6 p_2^{3r_2}}{27x^3(1-q_2)^{3r_2}}}}\right]\right\}.$$

تقريب نقطة السرج المشروط للتوزيعات الثنائية المركبة المبتورة

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المستخلص. تهدف غالبية الأبحاث المنشورة في موضوع المتغيرات العشوائية ثنائية المتغير المركبة إلى تفسير الظواهر في مختلف مجالات التطبيق. وعلى الرغم من أن هذه التوزيعات قد استحوذت على بعض الاهتمام في الدراسات السابقة، إلا أن هناك القليل من الأبحاث حول التوزيعات الثنائية المركبة بسبب صعوبة الحساب في استخدامها. تقدم هذه الدراسة طريقة تقريب نقطة السرج المشروط للتوزيعات الثنائية المركبة المبتورة، والتي أظهرت تفوق على طرق التقريب الأخرى في التوزيعات المتصلة والمنفصلة. وقد ناقشنا التقريب المشروط للدالة التوزيع التراكمي وقدمنا أمثلة على التوزيعات المتصلة والمنفصلة من توزيعات بواسون الثنائية المركبة المبتورة، وقارنا بين تقريب نقطة السرج المشروط والحساب المحكم للدالة التوزيع التراكمي حيث ظهر تفوق كبير لطريقة تقريب نقطة السرج المشروط.

الكلمات المفتاحية: تقريب نقطة السرج المشروط، التوزيع المركب ثنائي المتغير، دالة التوزيع التراكمي، التوزيع المركب ثنائي المتغير بواسون–جاما، التوزيع المركب ثنائي المتغير بواسون–سلبي ثنائي الحدين.